The Local Determination of Jordan Bases for *H*-Self-Adjoint Operators

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ABSTRACT

Let H be an invertible self-adjoint operator on a finite dimensional Hilbert space $\mathcal{X}$. A linear operator A is said to be H-self-adjoint (or self-adjoint relative to H) if $HA = A^*H$. Let $\sigma(A)$ denote, as usual, the spectrum of A. If A is H-self-adjoint, then A is similar to A^* and $\lambda \in \sigma(A)$ implies $\bar{\lambda} \in \sigma(A)$, so that the spectrum of A is symmetric with respect to the real axis. Given spectral information for A at an eigenvalue λ_0 ($\neq \bar{\lambda}_0$), we investigate the corresponding information at $\bar{\lambda}_0$ and, in particular, the unique pairing of Jordan bases for the root subspaces at λ_0 and $\bar{\lambda}_0$.

1. INTRODUCTION

Let $\mathcal{X}$ be a finite dimensional complex linear space with an inner product $(\ ,\ )$ and let H be a nonsingular self-adjoint linear operator on $\mathcal{X}$, i.e. $(Hx, y) = (x, Hy)$ for all $x, y \in \mathcal{X}$. We define an indefinite scalar product $[x, y]$ in $\mathcal{X}$ by

$$[x, y] = (Hx, y) \qquad \forall x, y \in \mathcal{X}.$$

This new product has all the properties of an inner product except that $[x, x]$ may be positive, negative, or zero for a nonzero vector x. The space $\mathcal{X}$ endowed with this product is called an indefinite scalar product space.

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Let A be a linear operator which is self-adjoint with respect to the indefinite scalar product. Thus, for any $x, y \in \mathcal{X}$

$$[Ax, y] = [x, Ay], \tag{1}$$

or, what is equivalent

$$HA = A^*H. \tag{2}$$

Such an operator will also be described as *H-self-adjoint.*

By virtue of the nonsingularity of H, (2) implies that A is similar to A^*. Hence $\sigma(A) = \sigma(A^*)$. If λ is an eigenvalue of A, we know that $\bar{\lambda}$ is an eigenvalue of A^*. Thus the fact that A is H-self-adjoint implies that the spectrum of A is symmetric with respect to the real axis. This symmetry is known to include the partial multiplicities of conjugate pairs $\lambda, \bar{\lambda}$ (see [3], for example).

Suppose now that $\mathcal{X}$ is a *real* linear space so that A and H have real matrix representations. Then there exists an invertible self-adjoint H (perhaps several) for which A is H-self-adjoint (see [3] or [4], for example). Furthermore, if we are given a Jordan basis $\{t_1, t_2, \dots, t_k\}$ for a root subspace $\mathcal{S}(\lambda)$ of A associated with eigenvalue λ $(\neq \bar{\lambda})$ [i.e. the kernel of $(I\lambda - A)^n$, where n is the dimension of $\mathcal{X}$], then a Jordan basis is obtained for the root subspace of $\bar{\lambda}$ just by taking the complex conjugates $\{\bar{t}_1, \dots, \bar{t}_k\}$; thus, $\mathcal{S}(\bar{\lambda}) = \overline{\mathcal{S}(\lambda)}$.

Given only that $HA = A^*H$, we certainly have the symmetry of the spectrum of A with respect to the real line, but in general $\mathcal{S}(\bar{\lambda}) \neq \overline{\mathcal{S}(\lambda)}$. In this note, it is our purpose to investigate the relationship between these subspaces. In particular, given a Jordan basis for $\mathcal{S}(\lambda)$, we explore ways and means of generating an appropriately orthonormalized Jordan basis for $\mathcal{S}(\bar{\lambda})$.

Subsequently, we identify a linear operator in $\mathcal{X}$ with its matrix representation in a suitable basis.

2. PRELIMINARIES

For an H-self-adjoint matrix A, a canonical reduction is possible in which A is reduced by similarity and H by congruence using one and the same matrix. When we have a description of this simplest canonical form then we can define more precisely the phrase "appropriately orthonormalised Jordan basis" used at the end of Section 2. We lead up to the complete canonical forms by noting some canonical examples. First, a $k \times k$ matrix whose (i, j) entry is 1, or 0, according as $i + j = k + 1$, or $i + j \neq k + 1$, will be called a *sip matrix* (standard involutory permutation).

Let $J(\lambda_0, k)$ denote a $k \times k$ Jordan block with *real* eigenvalue λ_0. If P_k is the $k \times k$ sip matrix, it is easily verified that $J(\lambda_0, k)$ is P_k-self-adjoint and also $(-P_k)$-self-adjoint.

If $\lambda_0 \neq \bar{\lambda}_0$ and $J(\lambda_0, k)$ is a corresponding Jordan block of size k, then the matrix

$$\begin{bmatrix} J(\lambda_0, k) & 0 \\ 0 & J(\bar{\lambda}_0, k) \end{bmatrix} \tag{3}$$

is self-adjoint in the indefinite scalar product defined by the invertible hermitian matrix

$$\begin{bmatrix} 0 & P_k \\ P_k & 0 \end{bmatrix}. \tag{4}$$

Now let A be an $n \times n$ matrix which is H-self-adjoint, and let J be a Jordan normal form for A. Since the spectrum of A is symmetric relative to the real axis, it may be assumed that J is a direct sum of Jordan blocks like $J(\lambda_0, k)$ when λ_0 is real, and matrices of type (3) associated with nonreal eigenvalues. With each *real* Jordan block we associate a positive or negative sign, and the ordered sequence of ± 1's, one for each real block, is denoted by ε.

As in the canonical examples, a signed sip matrix is associated with each real Jordan block, and a matrix of type (4) is associated with each pair of blocks in J of type (3). The direct sum of these "sip" blocks, ordered as in the blocks of J, is denoted by $P_{\varepsilon, J}$.

Then the following theorem on reduction of the pair (A, H) to the pair $(J, P_{\varepsilon, J})$ is quite well known (see [3], [4], and other sources mentioned there).

THEOREM 1. *Let A be H-self-adjoint. Then there exists a unique ε (up to permutation of signs in subsets corresponding to Jordan blocks of J having the same size and real eigenvalue) and an invertible T such that*

$$T^{-1}AT = J \quad \textit{and} \quad T^*HT = P_{\varepsilon, J}. \tag{5}$$

Since $AT = TJ$, it is clear that the columns of the matrix T form Jordan bases for the root subspaces of A. Furthermore, if λ is a nonreal eigenvalue with partial multiplicities $k_1, k_2, \dots, k_\alpha$ ($\kappa = \sum_{j=1}^{\alpha} k_j$), then the corresponding $\kappa \times \kappa$ submatrices of J and $P_{\varepsilon, J}$ are

$$J_\lambda = \operatorname{diag}[J(\lambda, k_1), J(\lambda, k_2), \dots, J(\lambda, k_\alpha)], \qquad P_\lambda = 0. \tag{6}$$

If A is $n\times n$ $(n \geqslant 2\kappa)$, the associated columns of T form an $n\times\kappa$ submatrix T_1, for which

$$AT_1 = T_1 J_\lambda \quad \text{and} \quad T_1^* H T_1 = 0. \tag{7}$$

Construct an $n\times\kappa$ submatrix T_2 of T associated in a similar way with eigenvalue $\bar{\lambda}$, and we have

$$AT_2 = T_2 J_{\bar{\lambda}} \quad \text{and} \quad T_2^* H T_2 = 0. \tag{8}$$

Furthermore

$$A\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}\begin{bmatrix} J_\lambda & 0 \\ 0 & J_{\bar{\lambda}} \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix}^* H \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & P_c \\ P_c & 0 \end{bmatrix}, \tag{10}$$

where

$$P_c = \operatorname{diag}\left[P_{k_1}, P_{k_2}, \dots, P_{k_\alpha} \right]. \tag{11}$$

We wish to emphasize that the orthonormalization of the Jordan bases for $\mathfrak{S}(\lambda)$ and $\mathfrak{S}(\bar{\lambda})$ required by Theorem 1 is seen from (10) to be expressed by

$$T_1^* H T_2 = P_c. \tag{12}$$

Note also that the subspaces $\mathfrak{S}(\lambda), \mathfrak{S}(\bar{\lambda})$ are necessarily isotropic (or H-neutral) and this is expressed by the relations $T_1^*HT_1 = 0$, $T_2^*HT_2 = 0$ of (7) and (8), which are also contained in (10).

Now it is known (Theorem 10.7 of [3]) that when A is real, there is a Jordan basis for $\mathfrak{S}(\lambda)$ represented by columns of matrix T, such that $T_2 = \bar{T}_1$ *and* the condition (12) still holds. This investigation began with the question: when A is not real, is there a natural choice of Jordan basis for $\mathfrak{S}(\lambda)$, i.e., of the matrix T_1, from which a T_2 is easily obtained (specializing to $T_2 = \bar{T}_1$ when A is real) and which satisfies (12)? Our conclusion is essentially negative but does provide new insights into the geometry of the root subspaces of A.

3. THE MAIN RESULTS

We now turn to the main objective of this paper, which is to investigate the relation between the root subspaces corresponding to the eigenvalues λ and $\bar{\lambda}$ of the H-self-adjoint operator A. We first establish some lemmas.

LEMMA 2. *Let A be an H-self-adjoint operator. Let λ_0 be an eigenvalue of A and*

$$P_{\lambda_0}(A) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\delta}(I\lambda - A)^{-1}\,d\lambda \tag{13}$$

denote the Riesz projector corresponding to λ_0. Then

$$P_{\bar{\lambda}_0}(A) = H^{-1}P^*_{\lambda_0}(A)H, \tag{14}$$

where $P^*_{\lambda_0}(A)$ is the adjoint of $P_{\lambda_0}(A)$.

Proof. In general, the Riesz projectors of A at λ_0 and of A^* at $\bar{\lambda}_0$ are related by

$$P^*_{\lambda_0}(A) = \left(P_{\lambda_0}(A)\right)^* = P_{\bar{\lambda}_0}(A^*)$$

(see Chapter 1 of [6], for example). Hence, using (2),

$$\begin{aligned} P^*_{\lambda_0}(A) = P_{\bar{\lambda}_0}(A^*) &= \frac{1}{2\pi i}\int_{|\lambda-\bar{\lambda}_0|=\delta}(I\lambda - A^*)^{-1}\,d\lambda \\ &= H\left[\frac{1}{2\pi i}\int_{|\lambda-\bar{\lambda}_0|=\delta}(I\lambda - A)^{-1}\,d\lambda\right]H^{-1} \\ &= HP_{\bar{\lambda}_0}(A)H^{-1}, \end{aligned}$$

from which (14) follows. ∎

Note that the range of the Riesz projector $P_{\lambda_0}(A)$ is the root subspace corresponding to λ_0, say $\mathfrak{S}(\lambda_0)$, and so given P_{λ_0}, $P_{\bar{\lambda}_0}$ can be calculated from (14) and hence $\mathfrak{S}(\bar{\lambda}_0)$. This result is, perhaps, rather deceptive in its simplicity. It does not help immediately in a constructive approach to the formation of orthonormalized Jordan bases. Furthermore, the Riesz projectors themselves are not easily computed. In Theorem 4 a more informative result will be obtained.

If $\mathcal{F}(\lambda_0)$ is the image of the Riesz projector for $\sigma(A)\backslash\{\lambda_0\}$, then $\mathbb{C}^n = \mathcal{E}(\lambda_0)\dot{+}\mathcal{F}(\lambda_0)$ and P_{λ_0} projects onto $\mathcal{E}(\lambda_0)$ along $\mathcal{F}(\lambda_0)$; i.e., for the image and kernel of P_{λ_0} we have $\operatorname{Im} P_{\lambda_0} = \mathcal{E}(\lambda_0)$ and $\operatorname{Ker} P(\lambda_0) = \mathcal{F}(\lambda_0)$. These remarks apply for *any* $n \times n$ matrix A, and indeed, if $\mathcal{F}(\lambda_0)^{\perp}$ denotes an orthogonal complement (in the usual inner product), we have

LEMMA 3. *For a general matrix A, the map $M = P^*_{\lambda_0}|_{\mathcal{E}(\lambda_0)}$ (the restriction of $P^*_{\lambda_0}$ to $\mathcal{E}(\lambda_0)$) defines a one-to-one map of $\mathcal{E}(\lambda_0)$ onto $\mathcal{F}(\lambda_0)^{\perp}$.*

Proof. We know that P_{λ_0} is onto $\mathcal{E}(\lambda_0)$ along $\mathcal{F}(\lambda_0)$. Hence $P^*_{\lambda_0}$ is the projection on $\mathcal{F}(\lambda_0)^{\perp}$ along $\mathcal{E}(\lambda_0)^{\perp}$. As the range of $P^*_{\lambda_0}P_{\lambda_0}$ is contained in the range of $P^*_{\lambda_0}$ and the rank of $P^*_{\lambda_0}P_{\lambda_0}$ is equal to the rank of $P^*_{\lambda_0}$, we have that the range of $P^*_{\lambda_0}P_{\lambda_0}$ equals the range of $P^*_{\lambda_0}$, namely $\mathcal{F}(\lambda_0)^{\perp}$. Hence, for any $x \in \mathcal{F}(\lambda_0)^{\perp}$ there is a y such that $x = P^*_{\lambda_0}P_{\lambda_0}y = P^*_{\lambda_0}z$, where $P_{\lambda_0}y = z \in \mathcal{E}(\lambda_0)$; i.e., every $x \in \mathcal{F}(\lambda_0)^{\perp}$ is the image of some $z \in \mathcal{E}(\lambda_0)$, as required.

Next, suppose $x_1, x_2 \in \mathcal{E}(\lambda_0)$ and let $P^*_{\lambda_0}x_1 = P^*_{\lambda_0}x_2$. Then $x_1 - x_2 \in \operatorname{Ker} P^*_{\lambda_0} = \mathcal{E}(\lambda_0)^{1}$. But $x_1 - x_2 \in \mathcal{E}(\lambda_0)$ and hence $x_1 = x_2$. ■

THEOREM 4. *If A is H-self-adjoint, then the map $H^{-1}P^*_{\lambda_0}|_{\mathcal{E}(\lambda_0)}$ is a one-to-one map of $\mathcal{E}(\lambda_0)$ onto $\mathcal{E}(\bar{\lambda}_0)$.*

Proof. By Lemma 3, $H^{-1}M$ is a one-to-one map of $\mathcal{E}(\lambda_0)$ onto $H^{-1}\mathcal{F}(\lambda_0)^{\perp}$. However, if A is H-self-adjoint, it can be shown that $H^{-1}\mathcal{F}(\lambda_0)^{\perp} = (H\mathcal{F}(\lambda_0))^{\perp} = \mathcal{E}(\bar{\lambda}_0)$, and hence the result. ■

Now we have the insight to formulate an explicit description of the transformation of a Jordan basis for $\mathcal{E}(\lambda_0)$ into one for $\mathcal{E}(\bar{\lambda}_0)$, and we emphasize that this is done using only *local* information, i.e., information depending only on the spectral properties of A at λ_0. An unsatisfactory (but apparently unavoidable) feature of the result is the continued presence of the projector P_{λ_0} in the construction. Theorem 4 is not used explicitly in the proof of Theorem 5, but its influence on the formulation of the theorem is apparent.

THEOREM 5. *Let A be H-self-adjoint, and let T_1 be an $n \times \kappa$ matrix whose columns form a Jordan basis for $\mathcal{E}(\lambda_0)$. Then the $n \times \kappa$ matrix*

$$T_2 = \left(H^{-1}P^*_{\lambda_0}T_1\right)S,$$

where $S=(T_1^*T_1)^{-1}P_c$, *has columns forming a Jordan basis for* $\mathfrak{S}(\bar{\lambda}_0)$. *Moreover,* T_1 *and* T_2 *satisfy the orthogonality condition* (12), *and* T_2 *is the unique matrix (with columns forming a Jordan basis for* $\mathfrak{S}(\bar{\lambda}_0)$*) for which* (12) *holds.*

Proof. It is known (Corollary 2.5 of [4], or [5]) that there is a Jordan basis for the root subspace of λ_0 with respect to A^T, whose members form the columns of an $n\times\kappa$ matrix Y, for which

$$P_{\lambda_0}=T_1P_cY^T \tag{15}$$

and, of course,

$$A^TY=YJ_{\lambda_0}. \tag{16}$$

Since P_{λ_0} is a projector, it follows from (15) that $Y^TT_1=P_c$.

With T_2 defined as in the theorem, we now show that $AT_2=T_2J_{\bar{\lambda}_0}$. From (15) it follows that

$$P_{\lambda_0}^*T_1S=\left(\bar{Y}P_cT_1^*\right)T_1(T_1^*T_1)^{-1}P_c=\bar{Y},$$

and consequently, using (16),

$$AT_2=A\left(H^{-1}P_{\lambda_0}^*T_1\right)S=H^{-1}A^*\bar{Y}=H^{-1}\bar{Y}J_{\bar{\lambda}_0}=\left(H^{-1}P_{\lambda_0}^*T_1S\right)J_{\bar{\lambda}_0}=T_2J_{\bar{\lambda}_0}.$$

For the orthogonality condition we have

$$T_1^*HT_2=T_1^*H\left(H^{-1}P_{\lambda_0}^*T_1\right)S=T_1^*\bar{Y}=P_c.$$

Finally, for the uniqueness, every matrix $\hat{T}_2$ whose columns form a Jordan basis for $\mathfrak{S}(\bar{\lambda}_0)$ has the form $\hat{T}_2=T_2U$ for some U in the class $\mathfrak{T}$ of all nonsingular matrices commuting with J_{λ_0}. Then

$$\hat{T}_2^*HT_1=U^*(T_2^*HT_1)=U^*P_c,$$

so if $\hat{T}_2$ also satisfies the orthogonality condition $\hat{T}_2^*HT_1=P_c$, we have $U^*P_c=P_c$ and, since $P_c^2=I$, $U^*=I$. Thus, $\hat{T}_2=T_2$. ■

The last theorem shows that given T_1 [determining $\mathfrak{S}(\lambda_0)$], there is a unique partner T_2 associated with $\mathfrak{S}(\bar{\lambda}_0)$ for which Equation (10), and hence

Theorem 1, will hold. In particular, if A is real there is no guarantee that T_2 will coincide with $\overline{T}_1$, even though the columns of $\overline{T}_1$ do form a Jordan basis for $\mathfrak{S}(\overline{\lambda}_0)$. It is known, however, that in this case there is a choice for T_1 for which $T_2 = \overline{T}_1$ (see Section 10.4 of [4]). We enlarge here on this point, taking advantage of a crucial lemma in [4]. The next theorem shows, in general, how pairs T_1, T_2 and $\tilde{T}_1, \tilde{T}_2$, say, are related.

As in the proof of the last theorem, $\mathfrak{T}$ denotes the group of invertible matrices which commute with J_{λ_0}.

THEOREM 6. *Let the columns of T_1, T_2 determine Jordan bases for $\mathfrak{S}(\lambda_0)$, $\mathfrak{S}(\overline{\lambda}_0)$ respectively, and let $T_1^* H T_2 = P_c$. Then*

(1) *any other pair of matrices $\tilde{T}_1, \tilde{T}_2$ with the same properties has the form*

$$\tilde{T}_1 = T_1 U, \quad \tilde{T}_2 = T_2\left(P_c U^{*-1} P_c\right), \qquad U \in \mathfrak{T}.$$

(2) *for any $V \in \mathfrak{T}$ there is a $U_V \in \mathfrak{T}$, depending only on V, such that*

$$\tilde{T}_1 = T_1 U_V, \qquad \tilde{T}_2 = T_2 \overline{V}\overline{U}_V \tag{17}$$

have the same properties as T_1, T_2.

Proof. (1): Any Jordan bases for $\mathfrak{S}(\lambda_0)$ and $\mathfrak{S}(\overline{\lambda}_0)$ can be generated by putting $\tilde{T}_1 = T_1 U$, $\tilde{T}_2 = T_2 V$, respectively, for some $U, V \in \mathfrak{T}$. Then

$$\tilde{T}_1 H \tilde{T}_2 = U^*(T_1^* H T_2) V = U^* P_c V.$$

Thus, the condition $\tilde{T}_1^* H \tilde{T}_2 = P_c$ is equivalent to $U^* P_c V = P_c$, or $V = P_c U^{*-1} P_c$ (and the composition on the right is, indeed, a member of $\mathfrak{T}$).

(2): Using Lemma 10.9 of [4], define $U_V \in \mathfrak{T}$ as a solution of

$$U_V^T (P_c V) U_V = P_c.$$

This implies that

$$\overline{V}\overline{U}_V = P_c U_V^{*-1} P_c,$$

and choosing $U = U_V$ in part (1), we get the result. ∎

Note that with A, H real there is a Jordan basis $\overline{T}_1$ in $\mathcal{E}(\overline{\lambda}_0)$. In this case V can be chosen so that $T_2\overline{V} = \overline{T}_1$, and then (17) gives

$$\tilde{T}_2 = \overline{T}_1\overline{U}_V = \overline{\tilde{T}}_1.$$

4. CONCLUDING REMARKS

Returning to our opening remarks, suppose that the root subspace $\mathcal{S}(\lambda_0)$ of the H-self-adjoint matrix A is prescribed, as above, by an $n \times \kappa$ matrix T_1. The associated J_{λ_0} is also known, and hence the matrix P_c of (11). How should one go about calculating T_2 without starting from scratch with the calculation of a Jordan basis for $\mathcal{S}(\overline{\lambda}_0)$? One possibility would be to find a full rank $n \times \kappa$ matrix $\hat{T}_2$ as a solution of the homogeneous matrix equation $A\hat{T}_2 - \hat{T}_2 J_{\overline{\lambda}_0} = 0$, thus generating an arbitrary Jordan basis for $\mathcal{S}(\overline{\lambda}_0)$ and avoiding any reference to the Riesz projector P_{λ_0} used in Theorem 5. Then form the matrix $U = P_c T_1^* H\hat{T}_2$, which, as is easily verified, is a member of $\mathfrak{T}$. Finally, let $T_2 = \hat{T}_2 U^{-1}$; it is found that $T_1^* H T_2 = P_c$. Thus, as described in Theorem 5, T_2 is the "partner" for T_1.

Theorem 6 can then be used to generate some preferred pair $\tilde{T}_1, \tilde{T}_2$, but just what this preference might be in the nonreal case must remain an open question.

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